# On cubic polynomials with a parabolic fixed point of a capture type(Complex Dynamics and its Related Fields)



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# On cubic polynomials with a parabolic fixed point of a capture type

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**Abstract**

We consider the location of each critical point of a cubic polynomial map with a parabolic fixed point. We show that, for any given number of iterations, there exists a cubic polynomial map with a parabolic fixed point such that the immediate parabolic basin contains just one of the critical points and the image of another critical point under the specified number of iterations.

## 1 Introduction

Let $f$ be any cubic polynomial. If $f$ has a parabolic fixed point $\alpha$, then a cycle of Fatou components of $f$ is called the immediate parabolic basin for $\alpha$ if the cycle contains a parabolic petal for $\alpha$.

Roughly speaking, in this note we consider the dynamically location of each critical points of $f$ with a parabolic fixed point whose basin contains both the critical points. We denote by $c_0$ and $c_1$ the critical points of $f$. Using the Haissinsky pinching deformation, we prove the following result:

**Theorem 1.1.** For any positive integer $n$, there exists a cubic polynomial map $f$ with a parabolic fixed point such that the immediate parabolic basin contains $c_0$ and $f^{\circ n}(c_1)$, and does not contain $f^{\circ k}(c_1)$ for any integer $k$ with $0 \leq k < n$.

Now, suppose that $f$ has a parabolic fixed point, and the parabolic basin contains $c_0$ and $c_1$. By analogy with Milnor [3], we shall define the types of this parabolic fixed point. For $j = 0, 1$, we denote by $U_j$ the Fatou component which contains $c_j$. Without loss of generality, we may assume that $U_0$ is contained in the immediate basin of the parabolic fixed point. Following from [3], there exist four possibilities as follows.

**Case 1:** The Fatou component is adjacent, i.e., $U_0 = U_1$.

**Case 2:** The Fatou component is bitransitive. Namely, $U_0 \neq U_1$, and moreover there exist the smallest positive integers $p, q > 0$ such that $f^{\circ p}(U_0) = U_1$ and $f^{\circ q}(U_1) = U_0$.

**Case 3:** The immediate parabolic basin captures $U_1$. Namely, the immediate parabolic basin does not contain $U_1$, but $f^{\circ k}(U_1)$ for some integer $k \geq 1$.

**Case 4:** Each of $U_0$ and $U_1$ is contained in the disjoint cycle of the immediate parabolic basin. Namely, $U_0$ and $U_1$ is contained in the immediate parabolic basin, and it follows that $f^{\circ n}(U_0) \cap f^{\circ m}(U_1) = \emptyset$ for any integers $n, m \geq 0$.

We define the types of the parabolic fixed point $\alpha$ as follows:

**Definition 1.2.** In Case 1, 2, 3 or 4, we say that $\alpha$ is a parabolic fixed point of an *adjacent*, *bitransitive*, *capture*, or *disjoint* type, respectively.

We will consider the type of the parabolic fixed point a the cubic polynomial map obtained by the Haissinsky pinching deformation, which is illustrated in the next section.

# 2 The Haissinsky Pinching deformation

Suppose that $f$ is any cubic polynomial map with an attracting fixed point $\alpha$. Let $B_f(\alpha)$ be the attracting basin for $\alpha$. We consider the Haissinsky pinching deformation of $f$ defined by pinching curves in $B_f(\alpha)$.

Following from [1], for any integer $q \geq 1$, there exist a smooth open arc $\gamma$ and a neighborhood $U \subset B_f(\alpha)$ of $\gamma$ satisfying the following conditions.

- $\overline{\gamma} \setminus \gamma$ consists of the attracting fixed point $\alpha$ and a repelling periodic point $\beta$ of period $q$.
- $f^{\circ q}(\gamma) = \gamma$, $f^{\circ q}(U) = U$, and $f^{\circ q}|_U$ is univalent.
- $f^{\circ n}(U) \cap f^{\circ m}(U) = \emptyset$ for any $0 \leq n < m < q$.
- There exist a number $\sigma > 0$ and a conformal map $\Phi_\sigma : U \to \{|z| < \pi\}$ such that $\Phi_\sigma \circ f^{\circ q}(z) = \Phi_\sigma(z) + \sigma$ for all $z \in U$.

We call the union $S := \bigcup_{k\geq 0} f^{\circ -k}(\overline{\gamma})$ the *support of pinching*, and define $S_0 := \bigcup_{k\geq 0} f^{\circ k}(\overline{\gamma})$. It follows from [1] that we have a sequence of quasiconformal maps $(h_t)_{t\geq 0}$ satisfying the following conditions.

- $h_t$ converges uniformly on $\widehat{\mathbb{C}}$ to a local quasiconformal map $h_\infty$ on $\widehat{\mathbb{C}}\setminus S$.
- $f_t := h_t \circ f \circ h_t^{-1}$ converges uniformly on $\widehat{\mathbb{C}}$ to a cubic polynomial $f_\infty$.
- $h_\infty(\alpha)$ is a parabolic fixed point of $f_\infty$.
- $h_\infty(S_0) = h_\infty(\alpha)$.

For further details, see [1] or [2].

# 3 Proof of Theorem 1.1

We first prove the following lemma needed later.

**Lemma 3.1.** Let $n$ be any positive integer, and let $\lambda$ be any complex number in $\mathbb{D}\setminus\{0\}$. Then there exists a cubic polynomial $f$, with $f^{\circ n}(c_1) = c_0$, such that $f$ has an attracting fixed point of multiplier $\lambda$ whose attracting basin is simply connected.

**Proof.** Consider a monic and centered cubic polynomial

$$P_{A,B}(z) = z^3 - 3Az + \sqrt{B},\ (A,B) \in \mathbb{C}^2.$$

Suppose that $P_{A,B}$ has a fixed point of multiplier $\lambda$. Then the fixed point is $\alpha_{A,\lambda} := \sqrt{A+\lambda/3}$, and hence, $P_{A,B}$ is affine conjugate to the cubic polynomial map

$$Q_{A,\lambda}(z) = z^3 + 3\alpha_{A,\lambda}z^2 + \lambda z$$

with critical points $c^{\pm}_{A,\lambda} := -\alpha_{A,\lambda} \pm \sqrt{A}$.

Suppose that $\lambda \in (-1,0)$, and the parameter $A$ is any real number $> -\lambda/3$ such that the attracting basin for zero is simply connected.

For each integer $k \geq 0$, we denote by $z_{A,\lambda}(k)$ the unique point on $\mathbb{R}_+$ such that $Q^{\circ k}_{A,\lambda}(z_{A,\lambda}(k)) = c^+_{A,\lambda}$. For any integer $k > 0$ and for any real number $A'$ with $A' > A$, we have $z_{A,\lambda}(k) < z_{A,\lambda}(k+1)$ and $z_{A,\lambda}(k) > z_{A',\lambda}(k)$. Thus since $Q_{A,\lambda}(c^-_{A,\lambda}) \to +\infty$ as $A \to +\infty$, for any integer $n > 0$ there exists a real number $A$ such that $Q^{\circ n}_{A,\lambda}(c^-_{A,\lambda}) = c^+_{A,\lambda}$.

Let $\lambda'$ be any complex number in $\mathbb{D}\setminus\{0\}$. Then it follows from [5] that there exists a quasiconformal map $h$ such that the cubic polynomial map $g := h \circ Q_{A,\lambda} \circ h^{-1}$ has an attracting fixed point with multiplier $\lambda'$. $\square$

We use the Haissinsky pinching deformation of $f$ obtained from this lemma.

**Proof of Theorem 1.1.** Without loss of generality, we may assume that $f(z) = z^3 + 3\alpha_{A,\lambda}z^2 + \lambda z$, $c_0 = c^+_{A,\lambda}$ and $c_1 = c^-_{A,\lambda}$.

Suppose that $\lambda$ is any real number with $-1 < \lambda < 0$, and $A$ is a real number $> -\lambda/3$ such that the attracting basin for zero is simply connected. Recall that $B_f(0)$ is the attracting basin for zero. Let $\varphi_f$ be the Kœnigs map such that $\varphi_f(0) = 0$, and $\varphi_f(z) = \lambda z$ for all $z \in B_f(0)$. We may assume that $\varphi_f(c_0) = 1$.

Define the half-line $\hat{\gamma} := i\mathbb{R}^+$, so that $\hat{\gamma}$ is periodic of period two under the iterates of the map $L(z) := \lambda z$. We denoted by $\gamma$ the connected component of the preimage of $\hat{\gamma}$ under $\varphi_f$ whose closure contains zero. Thus, we have the support of pinching $S := \bigcup_{k\geq 0} f^{\circ -k}(\overline{\gamma})$, and denote by $f_\infty$ the limit of the Haissinsky pinching deformation of $f$ defined by $S$.

Let $n$ be any positive integer. From Lemma 3.1, we have a parameter $A$ such that $f^{\circ n}(c_1) = c_0$. For each integer $k \geq 1$, we denote by $\alpha(k)$ the point on $\mathbb{R}_+$ such that $f^{\circ k}(\alpha(k)) = 0$, and by $S_{\alpha(k)}$ the connected component of $S$ which contains $\alpha(k)$.

At first consider the case $n \geq 2$. Since for each integer $k \geq 1$ the component $S_{\alpha(k)}$ separates the origin and $f^{\circ k}(c_1)$, it follows that $f_\infty$ has a parabolic fixed point of a capture type.

Next, consider the case $n = 1$. Since no connected component of $S$ separates the origin and $c_1$, it follows that $f_\infty$ has a parabolic fixed point of a bitransitive type.

In order to obtain a polynomial with a parabolic fixed point of a capture type, we will use the Branner-Hubbard deformation of $f$ obtained by wringing the almost complex structure on the attracting basin for zero (cf. [5]). In particular, we consider the Branner-Hubbard deformation which does not change the multiplier of the origin.

Let $s = 1 + 2\pi i/\log\lambda$, and let $l$ be the quasi-conformal map defined as $l(z) := z\,|z|^{s-1}$.

Recall that $\varphi_f$ is the Kœnigs map defined on $B_f(0)$. We define the holomorphic map $\psi_f : \mathbb{D} \to \mathbb{C}$ as the inverse map of $\varphi_f$ such that $\psi_f(0) = 0$.

Let $\sigma_0$ be the standard almost complex structure of $\widehat{\mathbb{C}}$, and let $\sigma$ be the almost complex structure defined as follows:

$$\sigma = \begin{cases} \sigma_0 & \text{on } \widehat{\mathbb{C}} \setminus B_f(0) \\ (l \circ \varphi_f)^*(\sigma_0) & \text{on } \psi_f(\mathbb{D}) \\ (l \circ \varphi_f \circ f^{\circ k})^*(\sigma_0) & \text{on } f^{-k}(\psi_f(\mathbb{D})) \setminus f^{-k+1}(\psi_f(\mathbb{D})), \end{cases} \tag{1}$$

where $k$ is an integer $\geq 1$.

From the Measurable Riemann Mapping Theorem, we obtain the quasi-conformal map $h$ such that $h^*\sigma_0 = \sigma$. Suppose that $h(0) = 0$, $h(1) = 1$

and $h(\infty) = \infty$. Then, we obtain a cubic polynomial map $g = h \circ f \circ h^{-1}$ with the attracting fixed point zero. It follows from [5] that the multiplier is $g'(0) = h(\lambda) = \lambda\,|\lambda|^{s-1} = \lambda$, and that the Kœnigs map $\varphi_g = l \circ \varphi_f \circ h^{-1}$.

Following from the argument similar to the above discussion, we define $S' \subset B_g(0)$ as the support of the pinching deformation, and denote by $g_\infty$ the limit of the pinching deformation of $g$ defined by the support $S'$.

There exists a cycle of connected components of $B_f(0) \setminus h^{-1}(S')$ under the iterates of $f$,

If $c_1$ is not contained in this cycle, then one of the critical points of $g_\infty$ is not contained in the immediate parabolic basin of $g_\infty$.

We consider the inverse image of $i\mathbb{R}$ under $\varphi \circ h^{-1}$. We introduce a preliminary definition as follows. For any point $z$ of the backward orbit of the origin, we denote by $D_f(z; r)$ the connected component of the set $\{w : |\varphi_f(w)| < r\}$ which contains the point $z$.

Since $f$ has no critical point in the open set $D_f(0; |\lambda|^{-1})$ except $c_0$, it follows that $f$ maps $D_f(0; |\lambda|^{-1}) \setminus \{c_0\}$ to $D_f(0; 1) \setminus \{c_0\}$ in two-to-one correspondence. Thus $f$ has the unique preimage $\alpha'$ of the origin such that $\alpha' \neq 0$ and $\alpha' \in D_f(0; |\lambda|^{-1}) \setminus \{c_0\}$.

We extend $\psi_f$ to the conformal map $\psi_{f,0}$ defined on $\mathbb{D}(0; |\lambda|^{-1}) \setminus [1, |\lambda|^{-1})$ to a subset of $D_f(0; |\lambda|^{-1})$. Moreover, we define $\psi_{f,1}$ as the conformal map defined on $\mathbb{D}(0; |\lambda|^{-1}) \setminus [1, |\lambda|^{-1})$ such that $\varphi_f \circ \psi_{f,1} \equiv$ identity map and $\psi_{f,1}(0) = \alpha'$.

The end points of the image of the set $\{yi \mid -|\lambda|^{-1} < y < |\lambda|^{-1}\}$ under $\psi_{f,0} \circ h^{-1}$ is contained in the boundary of $\psi_{f,1}(\mathbb{D}(0; |\lambda|^{-1}) \setminus [1, |\lambda|^{-1}))$. Hence, the connected component of the preimage of $i\mathbb{R}$ under $\varphi_f \circ h^{-1}$ which contains zero passes through the boundary of $\psi_{f,1}(\mathbb{D}(0; |\lambda|^{-1}) \setminus [1, |\lambda|^{-1}))$, and does not separate $c_0$ and $c_1$. On the other hand, the connected component of the preimage of $i\mathbb{R}$ under $\varphi_f \circ h^{-1}$ which contains $\alpha'$ separates $c_0$ and $c_1$. Therefore, the cycle of the Fatou components of $g$ does not contain one of the critical points of $g$, and hence $g_\infty$ has a parabolic fixed point of a capture type. □

# 4 Notes

Consider the family of cubic polynomials $P_{A,B}(z) := z^3 - 3Az + \sqrt{B}$ with $P_{A,B}(-\sqrt{A}) = \sqrt{A}$. We have $B = A(1-2A)^2$. The connectedness locus of the family of $P_{A,A(1-2A)^2}(z) = z^3 - 3Az + \sqrt{A} - 2A\sqrt{A}$, $A \in \mathbb{C}$, is showed in Figure 2.

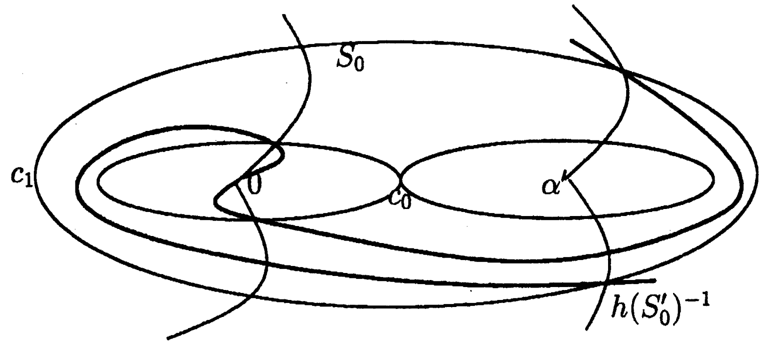


Figure 1: Sketch for the pinching curves.

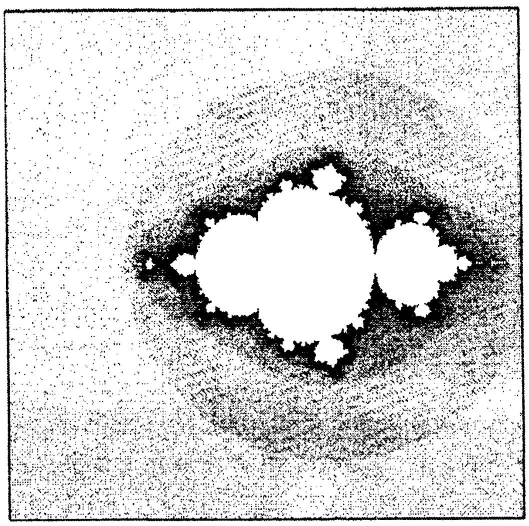

Figure 2: The connectedness locus of the family of cubic polynomials $P_{A,A(1-2A)^2}$, $A \in \mathbb{C}$.

$P_{A,A(1-2A)^2}$ is affine conjugate to the cubic polynomial map

$$F_A(z) := (P_{A,A(1-2A)^2}(\sqrt{A}z + \sqrt{A}) - \sqrt{A})/\sqrt{A} = Az^3 + 3Az^2 - 4A.$$

Suppose that $0 < |A| < 1/4$. Then the map $F_A$ satisfies the inequality $|F_A(z) + 4A| < |4A|$, that is, $F_A$ maps the disk of radius $|F_A(0)|$ centered at $F_A(0)$ into itself. Hence $F_A$ has an attracting fixed point in the disk.

Let $\alpha_A$ be the attracting fixed point.

**Proposition 4.1.** If $A$ turns around the origin once, then the multiplier of the attracting fixed point of $F_A$ turns around the origin twice.

**Proof.** Let $D$ be the disk of radius $|F_A(0)|$ centered at $F_A(0)$. If $A$ turns around the origin once, then the center of $D$ turns around the origin once.

Set $0 < r < 1/4$, $\theta \in [0, 1]$, and $A = re^{2\pi i\theta}$. Since the radius of $D$ is the constant $|F_A(0)|$, the attracting fixed point $\alpha_A$ also turns around the origin once. Thus the multiplier $F'_A(\alpha_A) = 3A\alpha_A(\alpha_A + 2)$ turns around the origin twice. $\square$

# References


[1] P. Haissinsky, *Pincement de polynômes*, Comment. Math. Helv. **77** (2002), no. 1, 1–23.

[2] T. Kawahira, *Note on dynamically stable perturbations of parabolics*, RIMS Kokyuroku, 1447 (2005), pp 90 - 107.

[3] J. Milnor, *Remarks on iterated cubic maps*, Experiment. Math. **1** (1992), no. 1, 5–24.

[4] J. Milnor, *Dynamics in one complex variable*, Friedr. Vieweg & Sohn, Braunschweig, 2000.

[5] C. Petersen, Tan Lei, *Branner-Hubbard motions and attracting dynamics*, preprint.

[6] L. Tan, *On pinching deformations of rational maps*, Ann. Sci. École Norm. Sup. (4) **35** (2002), no. 3, 353–370.